

REPRESENTATIONS OF ORBIFOLD GROUPOIDS

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ABSTRACT. Orbifold groupoids have been recently widely used to represent both effective and ineffective orbifolds. We show that every orbifold groupoid can be faithfully represented on a continuous family of finite dimensional Hilbert spaces. As a consequence we obtain the result that every orbifold groupoid is Morita equivalent to the translation groupoid of an almost free action of a proper bundle of topological groups.

1. INTRODUCTION

Orbifolds have generated a lot of interest in the recent mathematical and physical literature. As first defined in the paper of Satake [14], under the name of V -manifolds, they generalise the notion of smooth manifolds, by being slightly singular. More precisely, they are locally homeomorphic to the space of orbits of a finite group action on some Euclidian space. The original definition of Satake is equivalent to the modern definition of an effective orbifold.

The problem of generalising the definition of an orbifold to incorporate ineffective group actions in local charts is cumbersome. More convenient way is to use the language of Lie groupoids, as shown in the work of Moerdijk and Pronk [8, 11]. Although the theory of groupoids might seem abstract and lacking of geometric intuition at first, it provides a powerful tool to extend the differential geometric ideas to the (singular) spaces such as the spaces of leaves of a foliation [3, 4], spaces of orbits of Lie group actions and, in our case, orbifolds. Orbifold groupoids [1, 8, 9, 11] have been effectively used to represent orbifolds in the language of Lie groupoids. The space of orbits of such a groupoid carries a natural structure of an orbifold. Moreover, it is easy to describe effective orbifold groupoids as those groupoids that correspond to the effective orbifolds. In this way the definition of an ineffective orbifold comes for free in the framework of orbifold groupoids.

It is a well known result (see [1] or [9] for details) that the space of orbits of a smooth almost free action of a compact Lie group on a smooth manifold, such that the slice representations are effective, carries a natural structure of an effective orbifold. Conversely, each effective orbifold is isomorphic to the space of orbits of an almost free action of a unitary group on the bundle of frames of the orbifold. In the language of Lie groupoids this statement can be reformulated to saying that each effective orbifold is Morita equivalent to the translation groupoid of an almost free action of a compact group on a smooth manifold. It is conjectured (global quotient conjecture, see [1] for the formulation of the conjecture), but unknown at present, that similar statement holds for ineffective orbifolds as well. A partial result was obtained by Henriques and Metzler in [6], where they proved the statement for the class of orbifolds, whose ineffective groups have trivial centre.

The problem of presenting an orbifold groupoid as a translation groupoid of an almost free action of a compact Lie group is equivalent to finding a faithful unitary representation of the groupoid on some hermitian vector bundle over the space of

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objects of the groupoid. We show in this paper (Theorem 4.1) that such a representation (faithful and unitary) exists on a continuous family of finite dimensional Hilbert spaces over the space of objects of the groupoid. As a result (see Theorem 5.2) we can show that each orbifold groupoid is Morita equivalent to a translation groupoid of an almost free action of a proper bundle of topological groups, with each group being a finite product of unitary groups.

2. PRELIMINARIES

2.1. The Morita category of Lie groupoids. In this section we review the basic definitions and facts that will be used throughout the paper.

The notion of a topological groupoid is a combination and a generalization of both topological spaces and topological groups. The topological part is reflected in the space of orbits of the groupoid, which carries information of its transversal structure. On the other hand, the isotropy groups of the groupoid represent the algebraic part of the groupoid and make it a topological space with extra algebraic structure. Roughly, two groupoids represent the same geometric space if they have isomorphic transversal and algebraic structures. From the differential geometric viewpoint Lie groupoids form the most interesting class of topological groupoids and allow a natural extension of many of the operations on smooth manifolds.

For the convenience of the reader we first recall the notion of a topological groupoid (see [12] for more details) and proceed to the definition of Lie groupoids and generalised maps between them. Detailed exposition with many examples of Lie groupoids can be found in one of the books [7, 9, 10] and references cited there.

A topological groupoid G over the Hausdorff topological space G_0 is given by a structure of a category on the topological space G with objects G_0 , in which all arrows are invertible and all the structure maps

$$G \times_{G_0}^{s,t} G \xrightarrow{\text{mlt}} G \xrightarrow{\text{inv}} G \xrightleftharpoons[t]{s} G_0 \xrightarrow{\text{uni}} G$$

are continuous. The maps s, t and mlt are required to be open, while the map uni is an embedding. If $g \in G$ is any arrow with source $s(g) = x$ and target $t(g) = y$, and $g' \in G$ is another arrow with $s(g') = y$ and $t(g') = y'$, then the product $g'g = \text{mlt}(g', g)$ is an arrow from x to y' . The map uni assigns to each $x \in G_0$ the identity arrow $1_x = \text{uni}(x)$ in G , and we often identify G_0 with $\text{uni}(G_0)$. The map inv maps each $g \in G$ to its inverse g^{-1} . We use the notation $G(x, y) = s^{-1}(x) \cap t^{-1}(y)$ for the set of arrows from x to y and we denote by $G_x = G(x, x)$ the isotropy group of the element x .

Each groupoid G induces an equivalence relation on its space of objects G_0 by identifying two points if and only if there is an arrow between them. The resulting quotient map $q : G_0 \rightarrow G_0/G$ onto the space of orbits is an open surjection. The subset O of G_0 is G -invariant if it is saturated with respect to this natural equivalence relation. If $O \subset G_0$ is an open subset, the space $G|_O = t^{-1}(O) \cap s^{-1}(O)$ has a natural structure of a topological groupoid over O .

We say that the groupoid G is Hausdorff if the space of arrows G is a Hausdorff topological space. In this paper we will be mostly interested in proper topological groupoids. A topological groupoid G is proper if it is Hausdorff and if the map $(s, t) : G \rightarrow G_0 \times G_0$ is a proper continuous map.

A Lie groupoid is a topological groupoid G over G_0 , such that both G and G_0 are smooth manifolds and where all the structure maps are smooth. The maps s and t are required to be submersions with Hausdorff fibers, to insure the existence of a smooth manifold structure on the space $G \times_{G_0}^{s,t} G$, while the manifold of objects G_0 is usually taken to be Hausdorff and second countable. Here are some basic examples of topological groupoids.

Example 2.1. (i) Each smooth (Hausdorff, second countable) manifold M can be seen as a Lie groupoid with no nontrivial arrows, where $G = G_0 = M$ and where all the structure maps equal the identity map on the manifold M . On the other hand, each Lie group is a Lie groupoid with only one object and the structure maps induced from the Lie group structure.

(ii) Let a Lie group K act smoothly from the left on a smooth (Hausdorff, second countable) manifold M . The translation groupoid $K \ltimes M$ of this action has the manifold M as the space of objects and the space of arrows equal to $K \times M$. The source and target maps of the translation groupoid are given by the formulas $s(k, x) = x$ respectively $t(k, x) = k \cdot x$, while the multiplication is given by $(k', x')(k, x) = (k'k, x)$ for $x' = k \cdot x$. The identity and inverse maps are then induced from the group structure of the Lie group K . Translation groupoids associated to right actions of Lie groups on smooth manifolds can be defined analogously.

(iii) Let Q be a Hausdorff topological space. A bundle of topological groups over Q is given by a topological space U , together with an open surjection $r : U \rightarrow Q$, such that each fiber of the map r has a structure of a topological group and these structures vary continuously across Q . Each such bundle can be naturally seen as a topological groupoid $G = U$ over the space $G_0 = Q$ with the structure maps $s = t = r$ and the maps uni , mlt and inv induced by the group structures on the fibers of the map r . The bundle of topological groups U is locally trivial if the map r is locally trivial.

(iv) A bundle of topological groups U over Q is proper if it is proper as a topological groupoid. In this case each fiber is automatically a compact topological group. The converse is not always true. Let U be a trivial bundle of nontrivial finite groups over $\mathbb{R} - \{0\}$ together with the trivial group at $0 \in \mathbb{R}$. This bundle of groups over \mathbb{R} has compact fibers but it is not a proper bundle of topological groups.

(v) Let P be a Hausdorff topological space and let a bundle of topological groups $r : U \rightarrow Q$ act on P from the right along the map $\phi : P \rightarrow Q$ (see below for the definition of the groupoid action). The translation groupoid $P \rtimes U$ is a topological groupoid with the space of arrows $P \times_Q U$ over the space of objects P . The structure maps are given by: $t(p, u) = p$, $s(p, u) = p \cdot u$, $(p, u)(p', u') = (p, uu')$, $\text{uni}(p) = (p, 1_{\phi(p)})$ and $(p, u)^{-1} = (p \cdot u, u^{-1})$ for $\phi(p) = \phi(p') = r(u) = r(u')$ and $p \cdot u = p'$. If U is a proper bundle of topological groups it follows that $P \rtimes U$ is a proper topological groupoid.

Morphisms between Lie groupoids are smooth functors. Two Lie groupoids G and H are isomorphic if there exist morphisms $i : G \rightarrow H$ and $j : H \rightarrow G$ of Lie groupoids such that $j \circ i = \text{id}_G$ and $i \circ j = \text{id}_H$. However, in the context of the representation theory of groupoids the notion of a generalised morphism or a Hilsum-Skandalis map [10, 12], which we review in the sequel, is more suitable.

A smooth left action of a Lie groupoid G on a smooth manifold P along a smooth map $\pi : P \rightarrow G_0$ is a smooth map $\mu : G \times_{G_0}^{s, \pi} P \rightarrow P$, $(g, p) \mapsto g \cdot p$, which satisfies $\pi(g \cdot p) = t(g)$, $1_{\pi(p)} \cdot p = p$ and $g' \cdot (g \cdot p) = (g'g) \cdot p$, for all $g', g \in G$ and $p \in P$ with $s(g') = t(g)$ and $s(g) = \pi(p)$. We define right actions of Lie groupoids on smooth manifolds in a similar way.

Let G and H be Lie groupoids. A principal H -bundle over G is a smooth manifold P , equipped with a left action μ of G along a smooth submersion $\pi : P \rightarrow G_0$ and a right action η of H along a smooth map $\phi : P \rightarrow H_0$, such that (i) ϕ is G -invariant, π is H -invariant and both actions commute: $\phi(g \cdot p) = \phi(p)$, $\pi(p \cdot h) = \pi(p)$ and $g \cdot (p \cdot h) = (g \cdot p) \cdot h$ for every $g \in G$, $p \in P$ and $h \in H$ with $s(g) = \pi(p)$ and $\phi(p) = t(h)$, (ii) $\pi : P \rightarrow G_0$ is a principal right H -bundle: $(\text{pr}_1, \eta) : P \times_{H_0}^{\phi, t} H \rightarrow P \times_{G_0}^{\pi, \pi} P$ is a diffeomorphism.

A map $f: P \rightarrow P'$ between principal H -bundles P and P' over G is equivariant if it satisfies $\pi'(f(p)) = \pi(p)$, $\phi'(f(p)) = \phi(p)$ and $f(g \cdot p \cdot h) = g \cdot f(p) \cdot h$, for every $g \in G$, $p \in P$ and $h \in H$ with $s(g) = \pi(p)$ and $\phi(p) = t(h)$. Any such map is automatically a diffeomorphism. Principal H -bundles P and P' over G are isomorphic if there exists an equivariant diffeomorphism between them. A generalised map (sometimes called Hilsum-Skandalis map) from G to H is an isomorphism class of principal H -bundles over G .

If P is a principal H -bundle over G and P' is a principal K -bundle over H , for another Lie groupoid K , one can define the composition $P \otimes_H P'$ [10, 12, 13], which is a principal K -bundle over G . It is the quotient of $P \times_{H_0}^{\phi, \pi'} P'$ with respect to the diagonal action of the groupoid H . Lie groupoids form a category GPD [10, 12] with generalised maps from G to H as morphisms between groupoids G and H . A principal H -bundle P over G is called a Morita equivalence if it is also left G -principal. The isomorphisms in the category GPD correspond precisely to equivalence classes of Morita equivalences.

Actions of topological groupoids on topological spaces and the generalised maps between topological groupoids can be defined in a similar way. In the topological category all the maps are required to be continuous, while the condition that the projection map $\pi: P \rightarrow G_0$ of the principal H -bundle P is a surjective submersion is replaced by the condition that π is an open surjective map.

Orbifold groupoids, which are defined in the next subsection, are examples of Lie groupoids. In Section 4, where the representation theorem for orbifold groupoids (Theorem 4.1) is proven, we do not need the notion of a more general topological groupoid. However, presentation of an orbifold groupoid by a Morita equivalent translation groupoid (Theorem 5.2), associated to an almost free action of a proper bundle of topological groups, needs to be done in the topological category.

2.2. Orbifolds and Lie groupoids. Orbifolds are topological spaces which generalise the notion of smooth manifolds in a way that they locally look like quotients of smooth manifolds by a finite group action. They were first introduced by Satake in [14] under the name of V -manifolds. That original definition is equivalent to the definition of effective (also called reduced) orbifolds, found in the modern literature. A certain class of Lie groupoids, called orbifold groupoids [8, 9, 11], can be used to represent effective orbifolds and at the same time provide a way to define ineffective orbifolds.

Let Q be a topological space. An orbifold chart of dimension n on the space Q is given by a triple (\tilde{U}, G, ϕ) , where \tilde{U} is a connected open subset of \mathbb{R}^n , G is a finite subgroup of the group $\text{Diff}(\tilde{U})$ of smooth diffeomorphisms of \tilde{U} and $\phi: \tilde{U} \rightarrow Q$ is an open map that induces a homeomorphism between \tilde{U}/G and $U = \phi(\tilde{U})$. An embedding of an orbifold chart (\tilde{U}, G, ϕ) into an orbifold chart (\tilde{V}, H, ψ) is a smooth embedding $\lambda: \tilde{U} \rightarrow \tilde{V}$ that satisfies $\psi \circ \lambda = \phi$. The charts (\tilde{U}, G, ϕ) and (\tilde{V}, H, ψ) are compatible, if for any $z \in U \cap V$ there exists an orbifold chart (\tilde{W}, K, ν) with $z \in W$ and embeddings of the chart (\tilde{W}, K, ν) into the charts (\tilde{U}, G, ϕ) respectively (\tilde{V}, H, ψ) . An orbifold atlas (of dimension n) on Q is given by a family $\mathcal{U} = \{(U_i, G_i, \phi_i)\}_{i \in I}$ of pairwise compatible orbifold charts (of dimension n) that cover Q . An atlas \mathcal{U} refines the atlas \mathcal{V} if every chart of \mathcal{U} can be embedded into some chart of \mathcal{V} . Two orbifold atlases are equivalent if there exists an atlas that refines both of them. Effective orbifold of dimension n is a paracompact Hausdorff topological space Q together with an equivalence class of n -dimensional orbifold atlases on Q .

Primary examples of effective orbifolds are the orbit spaces of effective actions of finite groups on smooth manifolds, where the charts are given by connected

components of (small enough) invariant open subsets. More generally (see [1, 9]), let a compact Lie group K act smoothly and almost freely (with finite isotropy groups) on a smooth manifold M . Since the actions of compact Lie groups are proper, there exist local slices, equipped with the actions of the isotropy groups. If these actions are assumed to be effective, the slices can be used as the local orbifold charts on the space of orbits M/K .

The group actions, defined by the charts, and especially the isotropy groups form an important part of the orbifold structure. Namely, two orbifolds can be non-isomorphic, despite being homeomorphic, when seen as topological spaces. Since Lie groupoids have a natural built-in algebraic structure, they provide a suitable framework for the study of orbifolds [9, 11]. Furthermore, the notion of a generalised morphism between Lie groupoids representing orbifolds turns out to be the proper notion of a map between the corresponding orbifolds.

Let G be a Lie groupoid. If the maps s and t (and therefore all structure maps) are local diffeomorphisms we call G an étale Lie groupoid. A bisection of an étale Lie groupoid G is an open subset U of G such that both $s|_U$ and $t|_U$ are injective. Any such bisection U gives a local diffeomorphism $\tau_U : s(U) \rightarrow t(U)$, by $\tau_U = t|_U \circ (s|_U)^{-1}$. For any arrow $g \in G(x, y)$ there exists a bisection U_g containing g ; the germ at x of the induced local diffeomorphism is independent of the choice of the bisection.

Orbifold groupoid is a proper étale Lie groupoid. An orbifold groupoid G is effective if for each $x \in G_0$ and each nontrivial $g \in G_x$ the germ at x of some (and therefore every) local diffeomorphism τ_{U_g} , defined by a bisection through g , is nontrivial.

Crucial theorem in the connection between effective orbifolds and effective orbifold groupoids states that there is a natural structure of an effective orbifold [9, 11] on the space of orbits of an effective orbifold groupoid. In this way an ineffective orbifold groupoid can be seen as one possible way to define an ineffective orbifold.

2.3. Continuous families of Hilbert spaces. Representation theory of topological groupoids extends the classical representation theory of groups on vector spaces, where the latter are replaced by families of vector spaces, indexed by the space of objects of the groupoid. We first recall the definition and basic properties of a continuous family of Hilbert spaces over a topological space, as given in [5] (see also [2] for further examples).

Definition 2.2. *Let B be a locally compact Hausdorff topological space. A continuous family of Hilbert spaces over B is given by a pair $(\{E_x\}_{x \in B}, \Gamma)$, where E_x is a Hilbert space for each $x \in B$ and $\Gamma \subset \prod_{x \in B} E_x$ is a vector subspace that satisfies:*

- (1) *For each $x \in B$ and each $v \in E_x$ there exists $s \in \Gamma$ such that $s(x) = v$;*
- (2) *For every $s_1, s_2 \in \Gamma$ the function $x \mapsto \langle s_1(x), s_2(x) \rangle_x$ is a continuous function on B ;*
- (3) *If $w \in \prod_{x \in B} E_x$ satisfies: for each $x \in B$ and each $\epsilon > 0$ there exists a neighbourhood U of x and $s \in \Gamma$ such that $\|s(x') - w(x')\|_{x'} < \epsilon$ for all $x' \in U$, then $w \in \Gamma$.*

Family $(\{E_x\}_{x \in B}, \Gamma)$ is a continuous family of finite dimensional Hilbert spaces if all the Hilbert spaces E_x are finite dimensional.

From the topological viewpoint the following consequence of Definition 2.2 is useful and allows us to think of continuous families of Hilbert spaces as generalizations of hermitian vector bundles.

Proposition 2.3. *Let $(\{E_x\}_{x \in B}, \Gamma)$ be a continuous family of Hilbert spaces over a locally compact Hausdorff space B . Denote by $E = \coprod_{x \in B} E_x$ the disjoint union*

of the spaces $\{E_x\}_{x \in B}$. There exists a topology on the space E that makes the projection map $p : E \rightarrow B$ (which maps each Hilbert space E_x to the point x) a continuous open surjection and such that the space Γ equals the space of continuous sections of the map p .

Proof. We first define a basis for the topology on the total space E . For each open subset $V \subset B$, each $s \in \Gamma$ and each $\epsilon > 0$ define the tubular set $B(V, s, \epsilon) = \{v \in E \mid p(v) \in V, \|s(p(v)) - v\|_{p(v)} < \epsilon\}$. Condition (1) in Definition 2.2 insures that the family of all such tubular sets covers the space E . Now let $W_1 = B(V_1, s_1, \epsilon_1)$ and $W_2 = B(V_2, s_2, \epsilon_2)$ be two such tubular sets and choose arbitrary element $v \in W_1 \cap W_2$. For any such v the inequalities $\|s_i(p(v)) - v\|_{p(v)} < \epsilon_i$ hold for $i = 1, 2$ and there exists a section $s \in \Gamma$ such that $s(p(v)) = v$. Denote $\delta = \min\{\epsilon_1 - \|s_1(p(v)) - v\|_{p(v)}, \epsilon_2 - \|s_2(p(v)) - v\|_{p(v)}\}$. Since Γ is a vector subspace of $\prod_{x \in B} E_x$, $s_1 - s$ and $s_2 - s$ are elements of Γ as well. Using condition (2) in Definition 2.2 we can find open neighbourhoods U_1 and U_2 of the point $p(v)$ such that $\|s_i(x) - s(x)\|_x < \epsilon_i - \frac{\delta}{2}$ for $i = 1, 2$ and all $x \in U_1$ respectively $x \in U_2$. The tubular set $B(U_1 \cap U_2, s, \frac{\delta}{2})$ then satisfies $B(U_1 \cap U_2, s, \frac{\delta}{2}) \subset W_1 \cap W_2$ and contains the point v .

With the above topology the map p becomes a continuous open surjection. It remains to be proven that the space Γ equals the space of the continuous sections of the map p . Choose any section $s \in \Gamma$, sending $x \in B$ to $v \in E$. We want to show that s is a continuous section of the map p . For any basic open neighbourhood $B(V, s', \epsilon)$ of the element v we have $s - s' \in \Gamma$ and $\|s(x) - s'(x)\|_x < \epsilon$. Continuity of the map $y \mapsto \|s(y) - s'(y)\|_y$ gives us a neighbourhood U of the point x such that $\|s(y) - s'(y)\|_y < \epsilon$ on U . The neighbourhood $U \cap V$ then satisfies $s(U \cap V) \subset B(V, s', \epsilon)$, which proves that s is a continuous section of the map p . Conversely, let $s : B \rightarrow E$ be any continuous section of the map p . We will show that s satisfies condition (3) in Definition 2.2. Choose an element $x \in B$ and $\epsilon > 0$. By condition (1) in Definition 2.2 we can find $s' \in \Gamma$ such that $s(x) = s'(x)$. Since $s : B \rightarrow E$ is a continuous map, the set $V = s^{-1}(B(B, s', \epsilon))$ is an open neighbourhood of the point x such that $\|s(y) - s'(y)\|_y < \epsilon$ for every $y \in V$. The neighbourhood with this property exists for every $x \in B$ and every $\epsilon > 0$, therefore $s \in \Gamma$. \square

From now on we will denote the continuous family of Hilbert spaces $(\{E_x\}_{x \in B}, \Gamma)$ over B simply by E , according to the notations from the preceding proposition, and refer to Γ as the space of the continuous sections of the map p . It is not hard to check that the space Γ is in fact a module over the algebra of the continuous functions on the space B . The dimension $d(x)$ of the fiber E_x of a continuous family of Hilbert spaces is not necessarily constant along B , but it is a lower semi-continuous function on B , as can be seen by using properties (1) and (2) of Definition 2.2. Denote by $\text{supp}_o(E) = \{x \in B \mid d(x) > 0\}$ the support of the family of Hilbert spaces E . Notice that $\text{supp}_o(E)$ is an open subset of B since the dimension function d is lower semi-continuous.

Here are some examples of continuous families of Hilbert spaces that will be used later on in the paper.

Example 2.4. (i) Every n -dimensional hermitian vector bundle E over a locally compact Hausdorff space B is an example of a family of finite dimensional Hilbert spaces with fibers of constant dimension. Conversely, if the dimension of the fibers of the continuous family of Hilbert spaces E over B is a constant function on B , then E is actually a hermitian vector bundle over B .

(ii) Let B be a locally compact Hausdorff topological space, $O \subset B$ an open subset and E_O a hermitian vector bundle over the space O . The trivial extension

of the bundle E_O is the continuous family of Hilbert spaces E_O^B over B , defined as follows. The fiber of E_O^B over the point $x \in B$ is by definition

$$(E_O^B)_x = \begin{cases} (E_O)_x, & x \in O, \\ \{0\}, & \text{otherwise.} \end{cases}$$

We define the vector space $\Gamma(E_O^B)$ of sections of E_O^B to be the trivial extensions of those sections of E_O that tend to zero at the boundary of the space O in B . By definition, the section $s \in \Gamma(E_O)$ tends to zero at the boundary of the space O in B if for every $x \in \partial O$ and every $\epsilon > 0$ there exists a neighbourhood U of x in B such that $\|s(y)\|_y < \epsilon$ for all y in $U \cap O$. It is straightforward to check that the space $\Gamma(E_O^B)$ satisfies the conditions in Definition 2.2.

(iii) Let $\{E^i\}_{i \in I}$ be a collection of families of finite dimensional Hilbert spaces over the locally compact Hausdorff space B and assume that the family of open sets $\{\text{supp}_o(E^i)\}_{i \in I}$ is locally finite over B . The sum $E = \bigoplus_{i \in I} E^i$ of families $\{E^i\}_{i \in I}$ is defined as follows. First define $E_x = \bigoplus_{i \in I} E_x^i$ for each $x \in B$. Since the family $\{\text{supp}_o(E^i)\}_{i \in I}$ is locally finite, this sum is actually a finite sum, so E_x is a finite dimensional Hilbert space for every $x \in B$. The space of sections $\Gamma(E)$ is defined to be the product of the spaces $\Gamma(E^i)$. The induced topology on the space E coincides with the topology of the fibrewise product of the spaces E^i along the space B .

3. REPRESENTATIONS OF TOPOLOGICAL GROUPOIDS

Let G be a topological groupoid with locally compact space of objects G_0 and let $p : E \rightarrow G_0$ be a continuous family of Hilbert spaces over G_0 . A continuous representation of the groupoid G on the family E is given by a continuous left action of G on the space E , along the map p , such that each $g \in G(x, y)$ acts as a linear isomorphism $g : E_x \rightarrow E_y$. Representation of the groupoid G on the family of Hilbert spaces E is unitary if each $g \in G$ acts as a unitary map between the corresponding Hilbert spaces.

Example 3.1. (i) Let K be a topological group. Then the continuous representations of K , viewed as a topological groupoid, coincide with the continuous representations of K on Hilbert spaces. Each continuous family of Hilbert spaces over the locally compact space X is naturally a representation of the space X , seen as a topological groupoid.

(ii) Combining previous two examples we get the representations of the translation groupoid $K \ltimes X$ of a continuous action of the topological group K on the locally compact Hausdorff space X . These are precisely K -equivariant continuous families of Hilbert spaces over X , i.e. there is a fibrewise linear action of the group K on the total space E such that the projection map p is K -equivariant.

Generalised maps between groupoids can be used to pull back representations in the same sense as vector bundles can be pulled back by continuous maps. Let G and H be Lie groupoids and let P be a principal H -bundle over G . Assume that E is a hermitian vector bundle over H_0 , equipped with a unitary representation of the groupoid H , and denote by $\pi : P \rightarrow G_0$ respectively $\phi : P \rightarrow H_0$ the moment maps of the principal bundle P . The pull back bundle $\phi^*E = P \times_{H_0} E$ has a natural structure of a vector bundle over P with projection onto the first factor as the projection map. The groupoid H acts from the right on the space ϕ^*E by the formula: $(p, v) \cdot h = (p \cdot h, h^{-1} \cdot v)$. Since the action of H on P is along the fibers of the map π , it is easy to see that the map $\pi_G : \phi^*E/H \rightarrow G_0$, $\pi_G([p, v]) = \pi(p)$ is well defined and continuous. We will show that the space $P^*E = \phi^*E/H$ has a natural structure of a hermitian vector bundle over G_0 with projection map

π_G . Furthermore, the action of the groupoid G on the space P induces a unitary representation of the groupoid G on the bundle P^*E .

Proposition 3.2. *The space P^*E is a hermitian vector bundle over G_0 with a natural unitary representation of the Lie groupoid G .*

Proof. First consider the induced vector bundle ϕ^*E over P . Its fiber over the point $p \in P$ can be canonically identified to the fiber of the vector bundle E over the point $\phi(p) \in H_0$, while the formula $\langle (p, v_1), (p, v_2) \rangle_{\phi^*E} = \langle v_1, v_2 \rangle_E$ induces a scalar product on the fiber $(\phi^*E)_p$. So defined structures of Hilbert spaces on the fibers of the bundle ϕ^*E over P turn it into a hermitian vector bundle.

The groupoid H acts from the right on the space ϕ^*E by $(p, v) \cdot h = (p \cdot h, h^{-1} \cdot v)$ for $\phi(p) = t(h)$ and $v \in E_{t(h)}$. Let $r : \phi^*E \rightarrow G_0$ be the H -invariant projection defined with the formula $r((p, v)) = \pi(p)$ and denote by $\pi_G : \phi^*E/H \rightarrow G_0$ the induced map from the quotient space. First observe that the fibers of the map $r : \phi^*E \rightarrow G_0$ equal the restrictions $P^*E|_{\pi^{-1}(x)}$ of the bundle P^*E to the fibers of the map π over the points $x \in G_0$. Furthermore, since P is a principal H -bundle over G_0 , H acts freely and transitively along the fibers of the map π . Combining these two observations with the fact that the action of H on E is linear we get natural structures of Hilbert spaces on the fibers of the map π_G , which we now describe. Let $\delta = pr_2 \circ (pr_1, \eta)^{-1} : P \times_{G_0} P \rightarrow H$ be the continuous (actually smooth in our case) map, uniquely defined by the condition $p \cdot \delta(p, p') = p'$, for p and p' that satisfy $\pi(p) = \pi(p')$. Using the map δ we can define the maps $+ : P^*E \times_{G_0} P^*E \rightarrow P^*E$, $\cdot : \mathbb{C} \times P^*E \rightarrow P^*E$ and $\langle -, - \rangle_{P^*E} : P^*E \times_{G_0} P^*E \rightarrow \mathbb{C}$ by the formulas

$$[p, v] + [p', v'] = [p, v + \delta(p, p')v'],$$

$$\lambda[p, v] = [p, \lambda v],$$

$$\langle [p, v], [p', v'] \rangle_{P^*E} = \langle v, \delta(p, p')v' \rangle_E.$$

It is straightforward to verify that these maps are well defined, continuous and that they induce structures of Hilbert spaces on the fibers of the map π_G . To show that the map $\pi_G : P^*E \rightarrow G_0$ carries a structure of a vector bundle we have to show that it is locally trivial. Choose a point $x \in G_0$ and an element $p \in P$ with $\pi(p) = x$. Let $\psi : \phi^*E|_V \rightarrow V \times \mathbb{C}^n$ be a trivialization of the bundle ϕ^*E on the neighbourhood V of the point p . Since the map π is a submersion, there exists a local section $s : U \rightarrow V$ of the map $\pi|_V$, defined on some neighbourhood U of the point x , such that $s(x) = p$. Using the maps ψ and s we can define the trivialization $\psi' : P^*E|_U \rightarrow U \times \mathbb{C}^n$ by the formula $\psi'([p', v']) = (\pi(p'), pr_2(\psi(s(\pi(p'))), \delta(s(\pi(p')), p')v'))$. Finally, by defining $g \cdot [p, v] = [g \cdot p, v]$, we get a unitary representation of the Lie groupoid G on the hermitian vector bundle P^*E . \square

A representation of the groupoid G on the family of Hilbert spaces E is faithful if for each $x \in G_0$ the isotropy group G_x acts faithfully on the Hilbert space E_x . This condition is equivalent to the requirement that for each $x, y \in G_0$ and $g_1, g_2 \in G(x, y)$, with $g_1 \neq g_2$, the elements g_1 and g_2 induce different isomorphisms from E_x to E_y .

Proposition 3.3. *Let G and H be Lie groupoids and let P be a Morita equivalence between G and H . If the representation of the groupoid H on E is faithful the representation of the groupoid G on P^*E is faithful as well.*

Proof. We have to prove that for each $x \in G_0$ the isotropy group G_x acts faithfully on the vector space $(P^*E)_x$. Choose $x \in G_0$ and any $p \in \pi^{-1}(x) \subset P$. Since P is a Morita equivalence, the Lie groups G_x and $H_{\phi(p)}$ act freely and transitively on the space $P(p) = \pi^{-1}(x) \cap \phi^{-1}(\phi(p))$ from the left respectively from the right. Denote

by $i_p : G_x \rightarrow H_{\phi(p)}$ the induced bijection (which is in fact a group isomorphism), implicitly defined by the equation $g \cdot p = p \cdot i_p(g)$. Suppose that an arrow $g \in G_x$ acts as the identity transformation on the space $(P^*E)_x$, i.e. $g \cdot [p, v] = [p, v]$ for all $v \in E_{\phi(p)}$, where we have identified $[p, v] \in (P^*E)_x$ with $v \in E_{\phi(p)}$. Then the equality

$$[p, v] = g \cdot [p, v] = [g \cdot p, v] = [p \cdot i_p(g), v] = [p, i_p(g)v]$$

holds for all $v \in E_{\phi(p)}$. This shows that $i_p(g)$ acts as the identity on the space $E_{\phi(p)}$ and is therefore by the assumption of faithfulness of the representation of H on E equal to $1_{\phi(p)}$. Since i_p is a group isomorphism g must be equal to 1_x , which shows that the representation of G_x on $(P^*E)_x$ is faithful as well. \square

4. REPRESENTATIONS OF ORBIFOLD GROUPOIDS

The problem of representing an orbifold groupoid G faithfully and unitarily on a hermitian vector bundle is equivalent to finding a smooth almost free action of a compact Lie group K on a smooth manifold M such that the translation groupoid $M \rtimes K$ is Morita equivalent to the groupoid G .

A faithful unitary representation of the groupoid G on an n -dimensional hermitian vector bundle E over G_0 induces a free left action of the groupoid G on the principal $U(n)$ -bundle $\text{UFr}(E)$ of unitary frames of the bundle E , which commutes with the natural right action of the Lie group $U(n)$ on the bundle $\text{UFr}(E)$. Since the action of G on $\text{UFr}(E)$ is proper and free, the orbit space $G \backslash \text{UFr}(E)$ inherits a natural smooth structure. Moreover, since the actions of G and $U(n)$ on $\text{UFr}(E)$ commute, there exists an induced action of the group $U(n)$ on the manifold $G \backslash \text{UFr}(E)$, which is almost free as a consequence of the fact that G is an orbifold groupoid. It is then straightforward to check that G is Morita equivalent to the translation groupoid $(G \backslash \text{UFr}(E)) \rtimes U(n)$.

On the other hand, let the compact Lie group K act smoothly and almost freely from the right on the smooth manifold M and let P denote the Morita equivalence between the groupoids G and $M \rtimes K$. By the Peter-Weyl theorem for compact Lie groups there exists a finite dimensional Hilbert space V and a faithful unitary representation of the group K on V . This representation induces a faithful unitary representation of the groupoid $M \rtimes K$ on the trivial vector bundle $M \times V$, where the action is given by $(x, g) \cdot (x', v) = (x, g \cdot v)$ for $x' = x \cdot g$. Combining Propositions 3.2 and 3.3 we get a faithful unitary representation of the groupoid G on the hermitian vector bundle $P^*(M \times V)$ over G_0 .

The question whether every orbifold groupoid admits a faithful unitary representation on a hermitian vector bundle is believed to have a positive answer, but it is unproven at the moment. It has been long known to be true for effective orbifold groupoids, as sketched below. Each étale Lie groupoid G has a natural representation on the tangent bundle TG_0 , where an arrow $g \in G(x, y)$ acts via the differential of the local diffeomorphism τ_{U_g} , induced by some bisection U_g containing the arrow g . Straight from the definition it follows that this representation of the groupoid G on TG_0 is faithful if and only if G is an effective orbifold groupoid. This canonical representation can be extended to the representation of G on the complexified tangent bundle $T^{\mathbb{C}}G_0$ and made unitary by averaging an arbitrary hermitian metric on $T^{\mathbb{C}}G_0$. More recently, in the paper [6] by Henriques and Metzler, the authors proved the statement for the class of ineffective orbifold groupoids, whose ineffective isotropy groups have trivial centre.

However, in the broader framework of unitary representations on continuous families of finite dimensional Hilbert spaces we are able to prove the following result.

Theorem 4.1. *Let G be an orbifold groupoid over G_0 . Then there exists a faithful unitary representation of the groupoid G on a continuous family of finite dimensional Hilbert spaces over G_0 .*

We start by proving some propositions that will be needed in the proof of Theorem 4.1.

Proposition 4.2. *Let G be an orbifold groupoid over G_0 . For each $x \in G_0$ there exist a G -invariant open neighbourhood O_x of x and a faithful unitary representation of the groupoid $G|_{O_x}$ on a hermitian vector bundle E_{O_x} over O_x .*

Proof. In the proof of the proposition we use the following characterization of the local structure of orbifold groupoids [9, 11]. For each $x \in G_0$ there exist a neighbourhood U_x of x and a natural isomorphism of Lie groupoids $G|_{U_x} \cong G_x \ltimes U_x$, where each $g \in G_x$ acts on U_x by the diffeomorphism corresponding to the suitable bisection through g . Let $\mathbb{C}[G_x]$ denote the Hilbert space of complex functions on the finite group G_x with the orthonormal basis $\{\delta_g\}_{g \in G_x}$. The left regular representation of the group G_x on the space $\mathbb{C}[G_x]$ induces a faithful unitary representation of the groupoid $G|_{U_x} \cong G_x \ltimes U_x$ on the trivial vector bundle $U_x \times \mathbb{C}[G_x]$ by the formula $(g, x) \cdot (x, f) = (g \cdot x, g \cdot f)$.

The saturation $O_x = s(t^{-1}(U_x))$ of the open set U_x is again an open set since s is a submersion and hence an open map. It is straightforward to check that the manifold $P = t^{-1}(U_x)$, together with the left action of the groupoid $G|_{U_x}$ and the right action of the groupoid $G|_{O_x}$, defines a Morita equivalence between the groupoids $G|_{U_x}$ and $G|_{O_x}$. Denote by $E_{O_x} = (P^{-1})^*(U_x \times \mathbb{C}[G_x])$ the pullback bundle over O_x , together with the induced unitary representation of the groupoid $G|_{O_x}$. Since the representation of the groupoid $G|_{U_x}$ on $U_x \times \mathbb{C}[G_x]$ was faithful and since P^{-1} is a Morita equivalence the representation of $G|_{O_x}$ on E_{O_x} is faithful by Proposition 3.3. \square

Proposition 4.3. *Let G be an orbifold groupoid over G_0 and let $O \subset G_0$ be a G -invariant open subset of G_0 . Every unitary representation of the groupoid $G|_O$ on a hermitian vector bundle E_O over O can be extended to a unitary representation of the groupoid G on the continuous family of finite dimensional Hilbert spaces $E_O^{G_0}$ over G_0 .*

Proof. Let E_O be a Hermitian vector bundle over the space O , equipped with a unitary representation of the groupoid $G|_O$. Denote by $p : E_O^{G_0} \rightarrow G_0$ the trivial extension of the hermitian vector bundle E_O over O to a family of finite dimensional Hilbert spaces over G_0 as in Example 2.4. Recall that

$$(E_O^{G_0})_x = \begin{cases} (E_O)_x, & \text{if } x \in O, \\ \{0\}, & \text{otherwise.} \end{cases}$$

For any arrow $g \in G$ we define the action as follows:

- (i) If $g \in G|_O$ let g act on $E_O^{G_0}$ as it acts on E_O ;
- (ii) If $g \notin G|_O$ and $g \in G(x, y)$ then g acts in the only possible way, sending the vector 0_x to the vector 0_y .

This defines a unitary representation of the groupoid G on the family of finite dimensional Hilbert spaces $E_O^{G_0}$, which extends the representation of the groupoid $G|_O$ on the vector bundle E_O . To prove the claim of the proposition we have to check that this defines a continuous representation, i.e. the map $\mu : G \times_{G_0} E_O^{G_0} \rightarrow E_O^{G_0}$ is continuous.

First decompose the space G_0 as a disjoint union of G -invariant subspaces O , $V = \overline{O}^c$ and ∂O . Since O and V are open subsets of the space G_0 , the spaces $W_1 = G \times_{G_0} p^{-1}(V)$ respectively $W_2 = G \times_{G_0} p^{-1}(O)$ are open subspaces of the

space $G \times_{G_0} E_O^{G_0}$. Observing that $\mu|_{W_1}$ is basically the left action of the groupoid $G|_V$ on V , while $\mu|_{W_2}$ equals the action map of the representation of the groupoid $G|_O$ on the bundle E_O , we see that $\mu|_{W_1}$ respectively $\mu|_{W_2}$ are continuous maps. Now let $g \in G(x, y)$ be an arrow such that $x \in \partial O$ and therefore $y \in \partial O$. For such g there exists only one element in $G \times_{G_0} E_O^{G_0}$ with first coordinate g , namely $(g, 0_x)$ and we have $\mu(g, 0_x) = g \cdot 0_x = 0_y$. We need to show that the map μ is continuous at the point $(g, 0_x)$. To this extent choose arbitrary neighbourhood W of the point 0_y in $E_O^{G_0}$. By the definition of the topology on the space $E_O^{G_0}$ we can find a smaller tubular open neighbourhood $B(U_y, 0, \epsilon)$ of the point 0_y , where U_y is a neighbourhood of the point y in G_0 and 0 is the zero section of $E_O^{G_0}$. Shrinking the set U_y if necessary we can assume that there exists a bisection U of the groupoid G through the arrow g such that $t(U) = U_y$. The unitarity of the representation of G on $E_O^{G_0}$ now implies that $\mu(U \times_{G_0} B(s(U), 0, \epsilon)) \subset B(U_y, 0, \epsilon)$, which proves that μ is continuous at the point $(g, 0_x)$. \square

Proof of Theorem 4.1. Let G be an orbifold groupoid over G_0 . The quotient projection $q : G_0 \rightarrow G_0/G$ is an open surjective map, which insures that the space $Q = G_0/G$ is second countable and locally compact. Since G is a proper groupoid, the map $(s, t) : G \rightarrow G_0 \times G_0$ is a proper map between Hausdorff topological spaces and hence a closed map. This shows that $(s, t)(G) \subset G_0 \times G_0$ is a closed equivalence relation, so Q is a Hausdorff space. It follows that Q is paracompact.

We can use Proposition 4.2 to find for each $x \in G_0$ a G -invariant open neighbourhood O_x of the point x and a faithful unitary representation of the groupoid $G|_{O_x}$ on a Hermitian vector bundle E_{O_x} over O_x . The family $\{q(O_x)\}_{x \in G_0}$ is an open cover of the second countable paracompact space Q , so we can choose a countable, locally finite refinement $\{V'_i\}_{i \in \mathbb{N}}$ of the cover $\{q(O_x)\}_{x \in G_0}$. Pulling back the sets $\{V'_i\}_{i \in \mathbb{N}}$ to G_0 we get a locally finite covering $\{V_i\}_{i \in \mathbb{N}}$ of the space G_0 by G -invariant open subsets, where we denoted $V_i = q^{-1}(V'_i)$. For each $i \in \mathbb{N}$ we can choose some x_i , such that $V_i \subset O_{x_i}$, to get a faithful unitary representation of the groupoid $G|_{V_i}$ on the hermitian vector bundle $E_{V_i} = E_{O_{x_i}}|_{V_i}$. By Proposition 4.3 we can extend the unitary representation of the groupoid $G|_{V_i}$ on the bundle E_{V_i} to the unitary representation of the groupoid G on the family of finite dimensional Hilbert spaces $E^i = E_{V_i}^{G_0}$.

Let $E = \bigoplus_{i \in \mathbb{N}} E^i$ be the continuous family of finite dimensional Hilbert spaces over G_0 , defined as the sum of the families $\{E^i\}_{i \in \mathbb{N}}$ as in Example 2.4. The representations of the groupoid G on the families $\{E^i\}_{i \in \mathbb{N}}$ canonically induce a continuous unitary representation of the groupoid G on E , defined by $g \cdot (v_1, v_2, \dots) = (g \cdot v_1, g \cdot v_2, \dots)$. To see that the representation of G on E is faithful it is enough to show that for each $x \in G_0$ the group G_x acts faithfully on the Hilbert space E_x . Straight from the definition of the representation of G on E it follows that the representation of the group G_x on E_x decomposes as the direct sum of the representations of the group G_x on the spaces E_x^i for $i \in \mathbb{N}$. Since $\{V_i\}_{i \in \mathbb{N}}$ is a cover of the space G_0 , there exists some $i \in \mathbb{N}$ such that $x \in V_i$. Faithfulness of the representation of the groupoid $G|_{V_i}$ on the bundle E_{V_i} implies that the representation of the group G_x on E_x^i is faithful and consequently the representation of the group G_x on E_x is faithful as well. \square

5. ORBIFOLDS AS GLOBAL QUOTIENTS

5.1. Families of unitary frames and proper bundles of topological groups.

Let X be a locally compact Hausdorff space, $O \subset X$ an open subset and E_O an n -dimensional hermitian vector bundle over O . Denote by E_O^X the trivial extension of the hermitian vector bundle E_O to a continuous family of finite dimensional Hilbert

spaces over X as in Example 2.4. To the continuous family E_O^X of Hilbert spaces over X one can assign a family $\text{UFr}(E_O^X)$ of unitary frames over X as follows.

We first recall the definition of the principal $U(n)$ -bundle of unitary frames $\text{UFr}(E)$ of a hermitian vector bundle E over B . A unitary frame at a point $x \in B$ is an ordered orthonormal base of the Hilbert space E_x . We can represent it as a unitary isomorphism $e_x : \mathbb{C}^n \rightarrow E_x$, where \mathbb{C}^n is equipped with the standard scalar product. The set $\text{UFr}(E)_x$ of all frames of the bundle E at x is equipped with a natural right action of the Lie group $U(n)$: a group element $A \in U(n)$ acts on the frame $e_x \in \text{UFr}(E)_x$ by $e_x \cdot A = e_x \circ A$ to give a new frame at x . The bundle $\text{UFr}(E)$ of unitary frames of E is the disjoint union of all the spaces $\text{UFr}(E)_x$ with the natural projection map $\pi : \text{UFr}(E) \rightarrow B$, sending each of the sets $\text{UFr}(E)_x$ to their respective $x \in B$. A unitary local trivialisation $\phi_i : E|_{U_i} \rightarrow U_i \times \mathbb{C}^n$ of the hermitian vector bundle E induces a local trivialisation $\psi_i : \pi^{-1}(U_i) \rightarrow U_i \times U(n)$ of the bundle $\text{UFr}(E)$, given by $\psi_i(e) = (\pi(e), \phi_{i,\pi(e)} \circ e)$, where $\phi_{i,\pi(e)}$ is the unitary isomorphism from $E_{\pi(e)}$ to \mathbb{C}^n . The topology on the space $\text{UFr}(E)$ is the finest topology which makes all of the maps ψ_i^{-1} continuous.

The definition of the principal $U(n)$ -bundle $\text{UFr}(E_O)$ of unitary frames of the hermitian vector bundle E_O can be extended to define the family $\text{UFr}(E_O^X)$ of unitary frames of the trivial extension E_O^X of the bundle E_O . As a set $\text{UFr}(E_O^X)$ is defined to be the disjoint union

$$\text{UFr}(E_O^X) = \text{UFr}(E_O) \coprod (X \setminus O).$$

Let $\pi = \pi_{E_O} \coprod id|_{X \setminus O} : \text{UFr}(E_O^X) \rightarrow X$ denote the projection from the space $\text{UFr}(E_O^X)$ onto X , where $\pi_{E_O} : \text{UFr}(E_O) \rightarrow O$ is the ordinary projection from the bundle of the unitary frames of E_O onto O . We will define the topology on the space $\text{UFr}(E_O^X)$ by specifying its basis \mathcal{B} . The basic open sets of the space $\text{UFr}(E_O^X)$ are of two kinds:

- (1) For each open subset O' of X we have $\pi^{-1}(O') \in \mathcal{B}$;
- (2) If $O' \subset \text{UFr}(E_O)$ is an open subset then $O' \in \mathcal{B}$.

Equipped with the topology defined by the basis \mathcal{B} the family of frames $\text{UFr}(E_O^X)$ becomes a locally compact Hausdorff space such that the map π is a continuous open surjection.

Now let G be an orbifold groupoid over G_0 and let O be a G -invariant open subset of G_0 . To every n -dimensional hermitian vector bundle E_O over O we associate the proper bundle $U_O(n)$ of topological groups over the space $Q = G_0/G$ in the following way. The fiber of the bundle $U_O(n)$ at $x \in Q$ is given by

$$(U_O(n))_x = \begin{cases} U(n), & x \in q(O), \\ \{0\}, & \text{otherwise,} \end{cases}$$

where $q : G_0 \rightarrow Q$ is the quotient map. The topology on the space $U_O(n)$ is the quotient topology from the space $Q \times U(n)$, where the $U(n)$ -fibers are fibrewise shrunk to a point for the points outside of $q(O)$. The map $r_O : U_O(n) \rightarrow Q$ is a proper continuous map from which it follows that $U_O(n)$ is a proper bundle of topological groups.

We have a natural right action of the proper bundle of groups $U_O(n)$ on the family of unitary frames $\pi : \text{UFr}(E_O^{G_0}) \rightarrow G_0$ along the map $q \circ \pi : \text{UFr}(E_O^{G_0}) \rightarrow Q$. It is explicitly given by the formula $e_x \cdot A_{q(x)} = e_x \circ A_{q(x)}$, for $e_x \in \pi^{-1}(O)$ and where $A_{q(x)} \in U(n)$ is seen as a unitary isomorphism $A_{q(x)} : \mathbb{C}^n \rightarrow \mathbb{C}^n$. For x outside of O the action is defined in the only possible way. Note that the proper bundle of topological groups $U_O(n)$ acts freely and transitively along the fibers of the map π .

5.2. Presenting orbifolds as translation groupoids. Let M be a smooth manifold and K a compact Lie group acting smoothly and almost freely on the manifold M from the right. The translation groupoid $M \rtimes K$ is then Morita equivalent to an orbifold groupoid. The following proposition shows that the same is true if we replace the compact group K with some proper bundle of Lie groups U .

Proposition 5.1. *Let M be a smooth manifold and let U be a proper bundle of Lie groups over N , acting smoothly and almost freely from the right on the space M along the smooth map $\phi : M \rightarrow N$. Then the translation groupoid $M \rtimes U$ is Morita equivalent to an orbifold groupoid.*

Proof. The idea of the proof is similar to the case of an almost free action of a compact Lie group. A proper bundle of Lie groups U over N is a bundle of topological groups over N with a structure of a Lie groupoid. By definition the action of U on M is almost free if and only if the isotropy groups of the groupoid $M \rtimes U$ are finite and thus discrete. The groupoid $M \rtimes U$ is a proper Lie groupoid as a translation groupoid of a proper Lie groupoid. By Proposition 5.20 in [9] the groupoid $M \rtimes U$ is Morita equivalent to an étale Lie groupoid G . Since properness is invariant under Morita equivalence the groupoid G is proper and étale, thus an orbifold groupoid. \square

As proved in Theorem 4.1 each orbifold groupoid G admits a faithful unitary representation on a continuous family of finite dimensional Hilbert spaces over G_0 . We can use Theorem 4.1 to prove the following partial converse of Proposition 5.1.

Theorem 5.2. *Let G be an orbifold groupoid. Then G is Morita equivalent to a translation groupoid associated to a continuous almost free action of a proper bundle of topological groups on a topological space. The bundle can be chosen such that the fibers are finite products of unitary groups.*

We will prove Theorem 5.2 by constructing a space $\pi : \text{UFr}(E) \rightarrow G_0$ over G_0 , equipped with a free left action of the orbifold groupoid G along the map π and with a right action of a proper bundle of topological groups that acts freely and transitively along the fibers of the map π .

Proposition 5.3. *Let G be an orbifold groupoid over G_0 , O a G -invariant open subset of G_0 and let $E_O^{G_0}$ denote the trivial extension of the hermitian vector bundle E_O over O to a continuous family of finite dimensional Hilbert spaces over G_0 . Every continuous unitary representation of the groupoid G on the family $E_O^{G_0}$ induces a continuous action of the groupoid G on the family of frames $\text{UFr}(E_O^{G_0})$.*

Proof. Define the action of the groupoid G on the space $\text{UFr}(E_O^{G_0})$ as follows:

- (1) For $e_x \in \text{UFr}(E_O)_x$ and $g \in G(x, y)$ define $g \cdot e_x = g \circ e_x$, where g on the right is interpreted as a unitary map from E_x to E_y , coming from the representation of the groupoid $G|_O$ on the bundle E_O .
- (2) For $x \in G_0 \setminus O$ and $g \in G(x, y)$ define $g \cdot x = y$.

To show that this defines a continuous action $\mu : G \times_{G_0} \text{UFr}(E_O^{G_0}) \rightarrow \text{UFr}(E_O^{G_0})$ we use similar techniques as in the proof of Proposition 4.3. First decompose the space $\text{UFr}(E_O^{G_0})$ as a disjoint union of the subspaces $G_0 \setminus \overline{O}$, ∂O and $\text{UFr}(E_O)$.

The sets $G_0 \setminus \overline{O}$ and $\text{UFr}(E_O)$ are basic open subsets of the space $\text{UFr}(E_O^{G_0})$. First note that the restriction of the map μ to the open set $G \times_{G_0} (G_0 \setminus \overline{O})$ is equal to the natural left action of the groupoid G on $G_0 \setminus \overline{O}$ and thus continuous. Choose now any element $(g, e) \in G \times_{G_0} \text{UFr}(E_O)$, where $g \in G|_O$ is an arrow from x to y , and unitary local trivializations of the vector bundle E_O around x respectively y . The unitary representation of the groupoid G on E_O induces a continuous map m_g from

a small neighbourhood of the arrow g into the group $U(n)$, with respect to these two local trivializations. In the associated principal bundle charts the action of G on $\mathrm{UFr}(E_O)$ then looks like multiplication by the map m_g and is hence continuous.

It remains to be proven that μ is continuous at the points of the form $(g, x) \in G \times_{G_0} \mathrm{UFr}(E_{O^{G_0}}^{G_0})$ where $x \in \partial O \subset \mathrm{UFr}(E_{O^{G_0}}^{G_0})$ and $g \in G(x, y)$. We then have $g \cdot x = y$ and $y \in \partial O$ as well. Let W be any neighbourhood of the point $y \in \partial O \subset \mathrm{UFr}(E_{O^{G_0}}^{G_0})$. By the definition of the topology on the space $\mathrm{UFr}(E_{O^{G_0}}^{G_0})$ there exists a neighbourhood V of the point $y \in G_0$ such that $\pi^{-1}(V) \subset W$, where $\pi : \mathrm{UFr}(E_{O^{G_0}}^{G_0}) \rightarrow G_0$ is the projection map onto G_0 . Choose a bisection V_g of the arrow $g \in G$ such that $t(V_g) \subset V$. The set $V_g \times_{G_0} \mathrm{UFr}(E_{O^{G_0}}^{G_0})$ is then an open neighbourhood of the point (g, x) such that $\mu(V_g \times_{G_0} \mathrm{UFr}(E_{O^{G_0}}^{G_0})) \subset \pi^{-1}(V) \subset W$. \square

Now choose an orbifold groupoid G over G_0 and let E be a continuous family of finite dimensional Hilbert spaces over G_0 together with a faithful unitary representation of the groupoid G as constructed in the proof of Theorem 4.1. Here we use the same notations. The family E can be decomposed as a direct sum $E = \bigoplus_{i \in \mathbb{N}} E^i$, where each $E^i = E_{V_i}^{G_0}$ is a trivial extension of the hermitian vector bundle E_{V_i} over V_i . We define the family of unitary frames $\mathrm{UFr}(E)$ of the family E , with respect to the decomposition $E = \bigoplus_{i \in \mathbb{N}} E^i$, to be the fibrewise product of the families $\{\mathrm{UFr}(E^i)\}_{i \in \mathbb{N}}$ along the projection maps π_i ,

$$\mathrm{UFr}(E) = \{(e_1, e_2, \dots) \in \prod_{i \in \mathbb{N}} \mathrm{UFr}(E^i) \mid \pi_1(e_1) = \pi_2(e_2) = \dots\}.$$

The space $\mathrm{UFr}(E)$ has a natural projection π onto the space G_0 , induced from any of the projections π_i . The fiber of the space $\mathrm{UFr}(E)$ over a point $x \in G_0$ can be canonically identified with the product of the spaces of frames $\prod_{i \in \mathbb{N}} \mathrm{UFr}(E^i)_x$. Since the family $\{V_i\}_{i \in \mathbb{N}}$ is a locally finite cover of the space G_0 this product is in fact finite and therefore homeomorphic to a finite product of unitary groups.

Proposition 5.4. *The faithful unitary representation of the orbifold groupoid G on the continuous family of Hilbert spaces E over G_0 induces a continuous free action of the groupoid G on the family of unitary frames $\mathrm{UFr}(E)$.*

Proof. The action of the orbifold groupoid G on the space $\mathrm{UFr}(E)$ along the map π can be defined coordinatewise by $g \cdot (e_1, e_2, \dots) = (g \cdot e_1, g \cdot e_2, \dots)$. The continuity of this action follows from the fact that all the actions of the groupoid G on the spaces $\mathrm{UFr}(E^i)$ are continuous by Proposition 5.3 and from the fact that the topology on the space $\mathrm{UFr}(E)$ is the one induced from the product topology on $\prod_{i \in \mathbb{N}} \mathrm{UFr}(E^i)$.

Choose an arrow $g \in G$ and an element $e = (e_1, e_2, \dots) \in \mathrm{UFr}(E)_x$ such that $g \cdot e = e$. Then g must be an isotropy element, $g \in G_x$, and there exists some $i \in \mathbb{N}$ such that $x \in V_i$. From the definition of the bundle E^i it follows that G_x acts faithfully on E_x^i and therefore $g \cdot e_i = e_i$ implies $g = 1_x$. This shows that the action of the groupoid G on the space $\mathrm{UFr}(E)$ is free. \square

Denote by $U_i = U_{V_i}(n_i)$ the proper bundle of topological groups over the space $Q = G_0/G$, associated to the hermitian vector bundle E_{V_i} over the G -invariant open subset V_i of G_0 . The fibrewise product U of the bundles U_i has a natural structure of a proper bundle of topological groups over Q , with each fiber being isomorphic to a finite product of unitary groups. The right actions of the bundles U_i on the families of unitary frames $\mathrm{UFr}(E^i)$ induce a right action of the bundle U on the space $\mathrm{UFr}(E)$, defined by the formula $(e_1, e_2, \dots) \cdot (A_1, A_2, \dots) = (e_1 \cdot A_1, e_2 \cdot A_2, \dots)$. It is not hard to see that the proper bundle of groups U acts freely and transitively along the fibers of the map $\pi : \mathrm{UFr}(E) \rightarrow G_0$.

Proof of Theorem 5.2. For the convenience of the reader we first recall the data we have so far. Let G be an orbifold groupoid over G_0 and let $Q = G_0/G$ be the space of orbits of the groupoid G . We have constructed the space of frames $\text{UFr}(E)$, together with the moment maps $\pi : \text{UFr}(E) \rightarrow G_0$ (the projection map) and the map $u = q \circ \pi : \text{UFr}(E) \rightarrow Q$, where $q : G_0 \rightarrow Q$ is the quotient projection. There are actions of the groupoid G and of the proper bundle of groups $r : U \rightarrow Q$ on the space $\text{UFr}(E)$ from the left along the map π respectively from the right along the map u . Both of these actions are free and moreover U acts on the space $\text{UFr}(E)$ transitively along the fibers of the map π .

Now observe that both the actions are basically compositions of linear maps from the left respectively from the right. The associativity of the composition implies that the actions of G and U on the space $\text{UFr}(E)$ commute. Combining this with the fact that the map $u : \text{UFr}(E) \rightarrow Q$ is G -invariant, as shown by the equalities

$$u(g \cdot e) = q(\pi(g \cdot e)) = q(t(g)) = q(s(g)) = q(\pi(e)) = u(e),$$

we can define a right action of the proper bundle of groups U on the quotient space $G \backslash \text{UFr}(E)$, along the induced map $u' : G \backslash \text{UFr}(E) \rightarrow Q$, by the formula $[e] \cdot A = [e \cdot A]$ for $[e] \in G \backslash \text{UFr}(E)$ and $u'([e]) = r(A)$. This action is almost free since the action of U on $U(E)$ was free and since G has finite isotropy groups.

Let H be the (proper) translation groupoid associated to this action (see Example 2.1). It has the quotient $H_0 = G \backslash \text{UFr}(E)$ as the space of objects and the space of arrows equal to $(G \backslash \text{UFr}(E)) \times_Q U$. Note that H_0 is a Hausdorff space since $\text{UFr}(E)$ is Hausdorff and G is a proper groupoid. The multiplication in the groupoid H is defined by the formula $([e], A)([e'], A') = ([e], AA')$ for $[e \cdot A] = [e']$ and $u'([e]) = u'([e']) = r(A) = r(A')$. The source and the target maps of the groupoid H are given by $s([e], A) = [e \cdot A]$ respectively $t([e], A) = [e]$. We have a natural action of the translation groupoid H on the space $\text{UFr}(E)$, induced from the action of the proper bundle of groups U on $\text{UFr}(E)$ and defined by $e \cdot ([e], A) = e \cdot A$.

We will show that the space of frames $\text{UFr}(E)$, together with the moment maps $\pi : \text{UFr}(E) \rightarrow G_0$ and $\phi : \text{UFr}(E) \rightarrow H_0 = G \backslash \text{UFr}(E)$ (the quotient projection), and the actions of groupoids G respectively H , represents a Morita equivalence between the orbifold groupoid G and the translation groupoid H .

The translation groupoid H acts along the fibers of the map π because the bundle of groups U does so, while the groupoid G acts along the fibers of the map ϕ by the definition of ϕ . Similarly, it is not hard to see that both actions commute, so it remains to be proven that $\phi : \text{UFr}(E) \rightarrow H_0$ is a principal left G -bundle and that $\pi : \text{UFr}(E) \rightarrow G_0$ is a principal right H -bundle. Both the maps ϕ and π are open, the first being the quotient map of a groupoid action and the second one being open as a projection map of a fibrewise product along a family of open maps. Since the action of the groupoid G on the space $\text{UFr}(E)$ is free and transitive along the fibers of the map ϕ the map $i_G : G \times_{G_0} \text{UFr}(E) \rightarrow \text{UFr}(E) \times_{H_0} \text{UFr}(E)$, given by $i_G(g, e) = (g \cdot e, e)$, is a continuous bijection. Furthermore, since the groupoid G is proper, the action of G on $\text{UFr}(E)$ is proper so i_G is a closed map and hence a homeomorphism. This proves that $\phi : \text{UFr}(E) \rightarrow H_0$ is a principal left G -bundle. Similarly, the map $i_H : \text{UFr}(E) \times_{H_0} H \rightarrow \text{UFr}(E) \times_{G_0} \text{UFr}(E)$, defined by $i_H((e, ([e], A))) = (e, e \cdot A)$, defines a homeomorphism which shows that $\pi : \text{UFr}(E) \rightarrow G_0$ is a principal right H -bundle. \square

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